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A GENERALIZED POTENTIAL VORTICITY IN THE OCEAN

By

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Abstract

With the aid of Ertel's vorticity theorem [1942], a generalized potential vorticity for a compressible and baroclinic ocean is derived by examination of the thermodynamical property of potential density. The new potential vorticity $-\boldsymbol{W}\cdot\nabla\rho^*/\rho$ ($\boldsymbol{W}$ absolute vorticity, ρ^* potential density, ρ density) should be almost conserved under arbitrary conditions for the ocean except in the surface and bottom frictional boundary layers and mixed layers, and it is approximated by the product of the vertical gradient of ρ^* and the Coriolis parameter divided by ρ for deep water except in the vicinity of a strong current and the equator. Finally a method of its numerical calculation on a cross-current profile is shown. There is a negative or slightly positive potential vorticity in the vicinity of the current axis at about the depth of the main thermocline. In dealing with detailed structure in the vicinity of the current axis, the ordinary potential vorticity would ignore some terms of the same order of magnitude as itself in some limited regions with a relatively large value of anticyclonic horizontal shear. The Richardson's number might be a simplification of the ratio of a kind of the ordinary expression of potential vorticity to the additive term in the new potential vorticity which would be ignored in the ordinary one.

1. Introduction

The so-called potential vorticity Z/H is an important quantity for the inertial theory of the ocean circulation, where H is the thickness of a homogeneous layer of fluid and Z is the vertical component of absolute vorticity. This quantity, however, is defined only for a layer of fluid, and cannot be applied to points within that layer. Furthermore, it cannot be conserved under certain conditions even if the fluid is inviscid and immiscible. The main purpose of the present paper is to derive a general expression of potential vorticity which will be a conservative quantity defined for every point throughout most portions of the sea.

Let us start with Ertel's vorticity theorem for inviscid and immiscible fluids (Ertel [1942])[1],

[1] The author wishes to apologize to Prof. H. Ertel for having omitted this original paper in the reference of his paper (Kawai [1958]), where the same theorem was derived on use of a generalized natural coordinates.

$$\frac{D}{Dt}\left(\frac{\boldsymbol{W}\cdot\nabla\psi}{\rho}\right)=\frac{1}{\rho}\boldsymbol{W}\cdot\nabla\frac{D\psi}{Dt}+\frac{1}{\rho^3}\nabla\rho\times\nabla p\cdot\nabla\psi. \tag{1}$$

The symbols are defined as:

$\frac{D}{Dt}$ ——individual differentiation with respect to time t

∇ ——gradient operator

$\boldsymbol{W}$ ——absolute vorticity vector

ρ ——density

p ——pressure

ψ ——arbitrary scalar function.[1]

If ψ satisfies simultaneously two conditions

$$\frac{D\psi}{Dt}=0 \tag{2}$$

and

$$\nabla\psi=A\nabla p+B\nabla\rho, \tag{3}$$

where A and B are arbitrary scalar functions, both terms on the right-hand side of (1) vanish, meaning that $\boldsymbol{W}\cdot\nabla\psi/\rho$ is a conservative quantity.

Since the potential density and potential temperature are conservative quantities, as will be discussed in Section 2, they satisfy the condition (2). If sea water would have a constant salinity, they would exactly satisfy the condition (3) too, because its thermodynamical state could be specified by any couple of independent variables, for instance pressure and density only. We would then find out the quantity ψ which constitutes a new conservative quantity $\boldsymbol{W}\cdot\nabla\psi/\rho$.

The following discussion in Section 4 is simply a study of characteristics of gradients of potential density and potential temperature, and the conclusion will be reached that the potential density or, in some cases, potential temperature is practically eligible for the condition (3).

2. Individual conservations of salinity, potential temperature and potential density

The discussion in this Section is concerned with a compressible and baroclinic ocean without diffusion and viscosity. A water parcel in such an ocean undergoes neither heat nor salt nor water exchanges, not only between the

[1] As pointed out by Truesdell [1951], this theorem holds for any arbitrary vector or tensor function ψ, including the Lagrange-Beltrami vorticity equation as a special case.

parcel and its surroundings but between its internal parts. Although the parcel consists of infinitely many phases, interfacial tension is ignored throughout since its contribution to thermodynamical potentials may be relatively small.

According to Sverdrup *et al.* [1942], "by *conservative concentrations* are meant concentrations that are altered locally, except at the boundaries, by processes of diffusion and advection only". We shall, in the following, consider not only concentrations but physical properties or quantities, and we can say that their individual conservation is equivalent to the vanishing of their individual differentiation for an inviscid and immiscible water.

We are going to treat an "adiabatic process" of a water parcel under "adiabatic change" (Guggenheim [1949]), and it leads to individual conservations of entropy η and salinity S,

$$\frac{D\eta}{Dt}=0 \tag{4}$$

and

$$\frac{DS}{Dt}=0\ . \tag{5}$$

On use of the definition of the adiabatic lapse rate of temperature

$$\Gamma\equiv\left(\frac{\partial T}{\partial p}\right)_{\eta,S}\ , \tag{6}$$

the potential temperature θ referred to a certain constant pressure p_r is expressed by

and

$$\left.\begin{aligned}\theta&=T-\int_{p_r}^{p}\Gamma_a dp_a\\ \Gamma_a&\equiv\Gamma(S,\ T_a,\ p_a)\ ,\end{aligned}\right\} \tag{7}$$

where T_a is a value of temperature, corresponding to a value of pressure p_a at a point of state in the adiabatic process; and S, T and p are salinity, temperature and pressure *in situ* respectively. Partial differentiation of the first eq. of (7) with respect to p, keeping η and S constant, yields

$$\left(\frac{\partial\theta}{\partial p}\right)_{\eta,S}=\left(\frac{\partial T}{\partial p}\right)_{\eta,S}-\Gamma=0\ , \tag{8}$$

the last step being a consequence of the definition of Γ in (6). Equation (8) expresses that the adiabatic lapse rate of potential temperature is nought.

Specifying the thermodynamical state of sea water by three variables η, S and p, the individual change of potential temperature is expressed by

$$\frac{D\theta}{Dt}=\left(\frac{\partial\theta}{\partial\eta}\right)_{S,p}\frac{D\eta}{Dt}+\left(\frac{\partial\theta}{\partial S}\right)_{p,\eta}\frac{DS}{Dt}+\left(\frac{\partial\theta}{\partial p}\right)_{\eta,S}\frac{Dp}{Dt}=\left(\frac{\partial\theta}{\partial p}\right)_{\eta,S}\frac{Dp}{Dt}\ , \tag{9}$$

the last step being simply a consequence of the conservations of η and S shown in (4) and (5). Substituting (8) for (9), we obtain

$$\frac{D\theta}{Dt}=0\ , \tag{10}$$

expressing individual conservation of potential temperature.

The potential density ρ^* referred to a certain pressure p_r is defined by

$$\rho^*\equiv\rho(S,\theta,p_r)\ ,$$

and it follows that

$$\left(\frac{\partial\rho^*}{\partial p}\right)_{\eta,S}=\left(\frac{\partial\rho(S,\theta,p_r)}{\partial\theta}\right)_{p_r,S}\left(\frac{\partial\theta}{\partial p}\right)_{\eta,S}=0\ , \tag{11}$$

the last step being a consequence of (8). In the same way as in (9), we obtain

$$\frac{D\rho^*}{Dt}=\left(\frac{\partial\rho^*}{\partial p}\right)_{\eta,S}\frac{Dp}{Dt}\ . \tag{12}$$

Substitution of (11) for (12) leads to

$$\frac{D\rho^*}{Dt}=0\ , \tag{13}$$

expressing individual conservation of potential density.

We have now three conservative quantities S, θ and ρ^*, which satisfy the condition (2).

3. Gradients of salinity, potential temperature and potential density

Indicating components of the gradient operator normal and tangential to isobaric surface by ∇_n and ∇_t, gradients of the three conservative quantities are dissolved into two parts as follows.

$$\nabla S=\nabla_n S+\nabla_t S \tag{14a}$$

$$\nabla\theta=\nabla_n\theta+\nabla_t\theta \tag{14b}$$

$$\nabla\rho^*=\nabla_n\rho^*+\nabla_t\rho^* \tag{14c}$$

If the tangential gradient of a conservative quantity is negligibly smaller than the normal, the quantity will also satisfy the condition (3) because the gradient normal to isobaric surface is parallel to ∇p, and the ratio of the tangential term to the normal is almost equal to the inclination of isoline of the quantity in the profile in the direction of horizontal gradient of the quantity. Ignoring minute

Table 1. Ratios of the horizontal gradients of salinity, potential temperature and potential density to the vertical gradients for Sts. Atlantis I 5881 (11 April 1960, 37°40'N, 68°29'W) and 5882 (11 April 1960, 37°22'N, 68°32'W) $\Delta y = 33.34$ km

St.	5881—82	5881	5882	5881	5882	5881—82	5881	5882	5881	5882	5881—82	5881	5882	5881	5882
Depth m	$-\frac{\Delta S}{\Delta y}$ $\frac{10^{-10}‰}{cm}$	$\frac{\Delta S}{\Delta z}$ $\frac{10^{-7}‰}{cm}$		$-\frac{\Delta S}{\Delta y}\Big/\frac{\Delta S}{\Delta z}$ 10^{-3}		$-\frac{\Delta \theta_0}{\Delta y}$ $\frac{10^{-9}°C}{cm}$	$\frac{\Delta \theta_0}{\Delta z}$ $\frac{10^{-6}°C}{cm}$		$-\frac{\Delta \theta_0}{\Delta y}\Big/\frac{\Delta \theta_0}{\Delta z}$ 10^{-3}		$\frac{\Delta \rho_0^*}{\Delta y}$ $\frac{10^{-13}gr}{cm^4}$	$-\frac{\Delta \rho_0^*}{\Delta z}$ $\frac{10^{-10}gr}{cm^4}$		$-\frac{\Delta \rho_0^*}{\Delta y}\Big/\frac{\Delta \rho_0^*}{\Delta z}$ 10^{-3}	
300	99	80	7	1	14	−24	63	13	−0	−2	−141	96	38	−1	−4
400	324	120	22	3	15	126	70	8	2	16	60	78	3	1	20
500	774	232	162	3	5	387	137	98	3	4	327	143	113	2	3
750	1131	284	276	4	4	915	291	181	3	5	846	276	154	3	5
1000	534	52	132	10	4	924	84	204	11	5	876	60	208	15	4
1500	78	8	11	10	7	72	15	18	5	4	18	8	8	2	2
2000	−21	0	1	∞	−21	42	8	8	5	5	57	7	6	8	10
2500	9	1	2	9	5	45	9	8	5	6	33	7	7	5	5
3000	24	3	0	8	∞	66	10	9	7	7	36	7	8	5	5
3500	48	5	5	10	10	66	6	9	11	7	18	1	3	18	6
4000	21	1	2	21	11	24	2	4	12	6	3	1	1	3	3

thermohaline structures with scales less than a few tens of kilometers in a horizontal direction in deep water, the largest inclination of isotherms and isohalines in deep water among available data is found between Stations Atlantis I 5881 and 5882 from the survey "Gulf Stream '60" (Fuglister [1963]). The Stations are located in the vicinity of the Gulf Stream south of Georges Bank. As the Gulf Stream flows almost eastward in this case and properties of sea water are almost constant along the streamline, $|\nabla_t|$ may be given by $|\Delta/\Delta y|$, where positive y is toward the north.

Table 1 shows that the ratio of the second term to the first on the right-hand side of (14b) and (14c) is less than one per cent for θ and ρ^* except in the surface and bottom mixed layers (See values for depths of 300 m and 400 m at St. 5882 and for depths 3500 m and 4000 m at St. 5881.). A somewhat large value of the ratio for a depth of 1000 m at St. 5881 may be due to the difference in location between numerical calculations of horizontal and vertical gradients, because the temperature profile of the section (Fuglister [1963]) does not show steeper inclinations of isotherms at the point than in the neighborhood.

Since $\nabla_t\theta$ or $\nabla_t\rho^*$ has a component parallel to $\nabla\rho$ too, the component of $\nabla\theta$ or $\nabla\rho^*$ tangential to both of isobaric and isosteric surfaces will become much smaller than the value of $\nabla_t\theta$ or $\nabla_t\rho^*$ in Table 1. We may suppose that θ and ρ^* practically satisfy the condition (3), so far as their vertical gradients are not minute. Salinity profiles, however, show minimum or maximum layers frequently, and its vertical gradient becomes minute around the layers. This is the reason why salinity cannot satisfy the condition (3). In general, the potential density satisfies the condition (3) with better accuracy than the potential temperature, because the minimum or maximum layers of ρ^* are less frequently found than those of θ. A more detailed consideration will be given in Section 4.

4. Characteristics of the gradient of potential density and potential temperature

Describing the state of sea water by three independent variables ρ, p and S, we have

$$\nabla\rho^* = \left(\frac{\partial\rho^*}{\partial\rho}\right)_{S,p}\nabla\rho + \left(\frac{\partial\rho^*}{\partial p}\right)_{S,\rho}\nabla p + \left(\frac{\partial\rho^*}{\partial S}\right)_{\rho,p}\nabla S \tag{15a}$$

and

$$\nabla\theta = \left(\frac{\partial\theta}{\partial\rho}\right)_{S,p}\nabla\rho + \left(\frac{\partial\theta}{\partial p}\right)_{S,\rho}\nabla p + \left(\frac{\partial\theta}{\partial S}\right)_{\rho,p}\nabla S\ . \tag{15b}$$

Table 2. Thermodynamical derivatives for a typical water column in the Western Basin of the North Atlantic

			(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)
Depth	S	T	θ_0	$\left(\frac{\partial\rho}{\partial T}\right)_{S,p}$	$\left(\frac{\partial\rho}{\partial S}\right)_{T,p}$	$\left(\frac{\partial\theta_0}{\partial T}\right)_{S,p}$	$\left(\frac{\partial\theta_0}{\partial S}\right)_{T,p}$	$\left(\frac{\partial\rho_0^*}{\partial\theta_0}\right)_S$	$\left(\frac{\partial\rho_0^*}{\partial S}\right)_{\theta_0}$	$\left(\frac{\partial\theta_0}{\partial\rho}\right)_{S,p}$	$\left(\frac{\partial\theta_0}{\partial S}\right)_{\rho,p}$	$\left(\frac{\partial\rho_0^*}{\partial\rho}\right)_{S,p}$	$\left(\frac{\partial\rho_0^*}{\partial S}\right)_{\rho,p}$
m	‰	°C	°C	10^{-5}gr cm^3 °C	10^{-5}gr cm^3 ‰	1	10^{-2} °C ‰	10^{-5}gr cm^3 °C	10^{-5}gr cm^3 ‰	10^2 °C cm^3 gr	°C ‰	1	10^{-5}gr cm^3 ‰
0	—	—	—	—	—	1.000	0.000	—	—	—	—	1.00	0
1000	34.85	4.00	3.92	−13.1	77.9	0.992	−0.150	−10.5	79.5	−76	5.9	0.80	18
1500	—	—	—	−14.0	77.1	0.988	−0.226	−10.1	79.5	−71	5.4	0.71	25
2000	34.85	3.50	3.34	−14.8	76.4	0.984	−0.303	− 9.7	79.6	−67	5.1	0.65	30
2500	—	—	—	−15.6	75.7	0.980	−0.381	− 9.3	79.7	−63	4.8	0.58	36
3000	34.85	3.00	2.74	−16.4	75.1	0.977	−0.459	− 8.9	79.8	−60	4.5	0.53	40
3500	—	—	—	−17.2	74.5	0.973	−0.538	− 8.5	79.9	−57	4.2	0.48	44
4000	34.85	2.50	2.14	−18.0	73.9	0.970	−0.617	− 8.1	80.0	−54	4.0	0.44	48
5000	34.85	2.00	1.53	−19.7	72.6	0.963	−0.778	− 7.4	80.1	−49	3.5	0.36	54
6000	34.85	1.50	0.91	−21.1	71.5	0.956	−0.941	− 6.6	80.3	−45	3.2	0.30	59
7000	34.85	1.50	0.76	−23.0	70.3	0.951	−1.098	− 6.4	80.4	−41	2.9	0.26	62

(a) and (d) are calculated from Helland-Hansen's table [1930].
(b), (c), (f) and (g) are calculated from Hesselberg & Sverdrup's table [1914–15].
(e) is calculated from Fofonoff's empirical formula of potential temperature [1962].
(h)=(d)/(b), (i)=(e)−(k)(c), (j)=(f)(h), (k)=(g)+(f)(i); See eqs. (18a, b, c, d) in the text.

On use of thermodynamical relationships[1]

$$\left(\frac{\partial\rho^*}{\partial p}\right)_{S,\rho}=-\frac{1}{c^2}\left(\frac{\partial\rho^*}{\partial\rho}\right)_{S,p} \tag{16a}$$

and

$$\left(\frac{\partial\theta}{\partial p}\right)_{S,\rho}=-\frac{1}{c^2}\left(\frac{\partial\theta}{\partial\rho}\right)_{S,p}, \tag{16b}$$

eqs. (15a) and (15b) become

$$\nabla\rho^*=\left(\frac{\partial\rho^*}{\partial\rho}\right)_{S,p}\left(\nabla\rho-\frac{\nabla p}{c^2}\right)+\left(\frac{\partial\rho^*}{\partial S}\right)_{\rho,p}\nabla S \tag{17a}$$

and

$$\nabla\theta=\left(\frac{\partial\theta}{\partial\rho}\right)_{S,p}\left(\nabla\rho-\frac{\nabla p}{c^2}\right)+\left(\frac{\partial\theta}{\partial S}\right)_{\rho,p}\nabla S, \tag{17b}$$

where c indicates the sound speed in the sea.

Values of four thermodynamical derivatives in (17a) and (17b) for a typical water column are calculated in Table 2, where the pressure of one atmosphere is adopted as the reference pressure of potential temperature and potential density, being specified by the subscript $_0$ as in $\rho_0{}^*$. The calculation is based on the following relationships.

$$\left(\frac{\partial\theta}{\partial\rho}\right)_{S,p}=\frac{\left(\frac{\partial\theta}{\partial T}\right)_{S,p}}{\left(\frac{\partial\rho}{\partial T}\right)_{S,p}} \tag{18a}$$

$$\left(\frac{\partial\theta}{\partial S}\right)_{\rho,p}=\left(\frac{\partial\theta}{\partial S}\right)_{T,p}-\left(\frac{\partial\theta}{\partial\rho}\right)_{S,p}\left(\frac{\partial\rho}{\partial S}\right)_{T,p} \tag{18b}$$

$$\left(\frac{\partial\rho^*}{\partial\rho}\right)_{S,p}=\left(\frac{\partial\rho^*}{\partial\theta}\right)_{S}\left(\frac{\partial\theta}{\partial\rho}\right)_{S,p} \tag{18c}$$

$$\left(\frac{\partial\rho^*}{\partial S}\right)_{\rho,p}=\left(\frac{\partial\rho^*}{\partial S}\right)_{\theta}+\left(\frac{\partial\rho^*}{\partial\theta}\right)_{S}\left(\frac{\partial\theta}{\partial S}\right)_{\rho,p} \tag{18d}$$

The vertical components of (17a) and (17b) are expressed by

$$\frac{\partial\rho^*}{\partial z}=\left(\frac{\partial\rho^*}{\partial\rho}\right)_{S,p}\left(\frac{\partial\rho}{\partial z}+\frac{\rho g}{c^2}\right)+\left(\frac{\partial\rho^*}{\partial S}\right)_{\rho,p}\frac{\partial S}{\partial z} \tag{19a}$$

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[1] The relationships (16a) and (16b) are consequences of

$$\left(\frac{\partial\rho^*}{\partial p}\right)_{\eta,s}=\left(\frac{\partial\rho^*}{\partial\rho}\right)_{s,p}\left(\frac{\partial\rho}{\partial p}\right)_{\eta,s}+\left(\frac{\partial\rho^*}{\partial p}\right)_{s,\rho},$$

$$\left(\frac{\partial\theta}{\partial p}\right)_{\eta,s}=\left(\frac{\partial\theta}{\partial\rho}\right)_{s,\eta}\left(\frac{\partial\rho}{\partial p}\right)_{\eta,s}+\left(\frac{\partial\theta}{\partial p}\right)_{s,\rho},$$

$$\frac{1}{c^2}=\left(\frac{\partial\rho}{\partial p}\right)_{\eta,s}$$

and eqs. (11) and (8).

and

$$\frac{\partial\theta}{\partial z}=\left(\frac{\partial\theta}{\partial\rho}\right)_{S,p}\left(\frac{\partial\rho}{\partial z}+\frac{\rho g}{c^2}\right)+\left(\frac{\partial\theta}{\partial S}\right)_{\rho,p}\frac{\partial S}{\partial z}\ , \tag{19b}$$

where z is positive upward and $-(\partial\rho/\partial z+\rho g/c^2)$ is nothing but the vertical stability (Pollak [1954]). Accordingly, the vector $(\nabla\rho-\nabla p/c^2)$ may be called "stability vector". Most portions of the sea have a stable stratification, namely—

$$-\left(\frac{\partial\rho}{\partial z}+\frac{\rho g}{c^2}\right)>0\ , \tag{20}$$

and it follows that if either

$$\left(\frac{\partial\rho^*}{\partial z}\right)^2>\left[\left(\frac{\partial\rho^*}{\partial S}\right)_{\rho,p}\frac{\partial S}{\partial z}\right]^2 \tag{21a}$$

or

$$\left(\frac{\partial\theta}{\partial z}\right)^2>\left[\left(\frac{\partial\theta}{\partial S}\right)_{\rho,p}\frac{\partial S}{\partial z}\right]^2 \tag{21b}$$

is satisfied, $\partial\rho^*/\partial z$ or $-\partial\theta/\partial z$ should be negative by a simple algebra, because $(\partial\rho^*/\partial\rho)_{S,p}$ is positive and $(\partial\theta/\partial\rho)_{S,p}$ is negative, as seen in Table 2.

Table 3. Ratios between two terms in eqs. (17a) and (17b)

St.	5881	5882	5881	5882
Depth	$\left(\frac{\partial\rho_0^*}{\partial S}\right)_{\rho,p}\frac{\Delta S}{\Delta z}\Big/-\frac{\Delta\rho_0^*}{\Delta z}$		$\left(\frac{\partial\theta_0}{\partial S}\right)_{\rho,p}\frac{\Delta S}{\Delta z}\Big/\frac{\Delta\theta_0}{\Delta z}$	
m	1		1	
1000	0.2	0.1	0.4	0.4
1500	0.3	0.3	0.3	0.3
2000	0.0	0.1	0.0	0.1
2500	0.1	0.1	0.1	0.1
3000	0.2	0.0	0.1	0.0
3500	2.2	0.7	0.4	0.2
4000	0.5	1.0	0.2	0.2

Table 3, which is calculated with the values in Tables 1 and 2, shows that the condition (21a) or (21b) is satisfied except in the mixed layers, and it follows, based on the discussion above, that ρ^* or θ scarcely has inversion layers.

The characteristics of $\nabla\rho^*$ and $\nabla\theta$ discussed above come from a relative smallness of salinity gradient and the stable stratification.

The result in Table 3 is, however, based on a water column in the Western Basin of the North Atlantic. Since $\partial S/\partial z$ is positive or slightly negative in deep water in the Basin, $\partial\theta/\partial z$ becomes positive by (19b) as far as the stratification is stable, but in general, $\partial\theta/\partial z$ does not always become positive for a large value of negative $\partial S/\partial z$. As estimated from Table 2, the derivative $(\partial\rho^*/\partial S)_{\rho,p}$ approaches zero as pressure *in situ* approaches the reference pressure p_r, while $(\partial\theta/\partial S)_{\rho,p}$ does not. At a depth with the reference pressure, eq. (17a) becomes

$$(\nabla\rho^*)_{p_r}=\left(\nabla\rho-\frac{\nabla p}{c^2}\right)_{p_r},$$

while eq. (17b) becomes

$$(\nabla\theta)_{p_r}=\left(\frac{\partial T}{\partial\rho}\right)_{S,p_r}\left(\nabla\rho-\frac{\nabla p}{c^2}\right)_{p_r}+\left(\frac{\partial T}{\partial S}\right)_{\rho,p_r}(\nabla S)_{p_r},$$

where the last term does not vanish generally. So far as the stability vector has a downward component, $\nabla\rho^*$ has a downward component too in the vicinity of the reference pressure, while $\nabla\theta$ does not necessarily have an upward component. This is the reason why ρ^* has less frequently inversion layers and core layers than θ. Thus we may conclude that the potential density is the best one which satisfies the condition (3) among the three conservative quantities.

No maximum of ρ_0^* is found for these Stations, but another example shows a maximum located at a depth of 4500 m (Fig. 1). A layer of the maximum is clearly found throughout the bottom water of the Western Basin of the North Atlantic, when the depth of the bottom exceeds 4500 m. Since $\partial\rho_0^*/\partial z$ is minute in the vicinity of the layer, ρ_0^* cannot satisfy the condition (3) for the layer. Using a pressure of 5000 decibars as the reference, however, there is no significant maximum value of ρ_5^* in the bottom water (Fig. 1), the subscript $_5$ being the indicator of 5000 decibars. In this case, the condition (3) holds for the bottom water only except the bottom mixed layer.

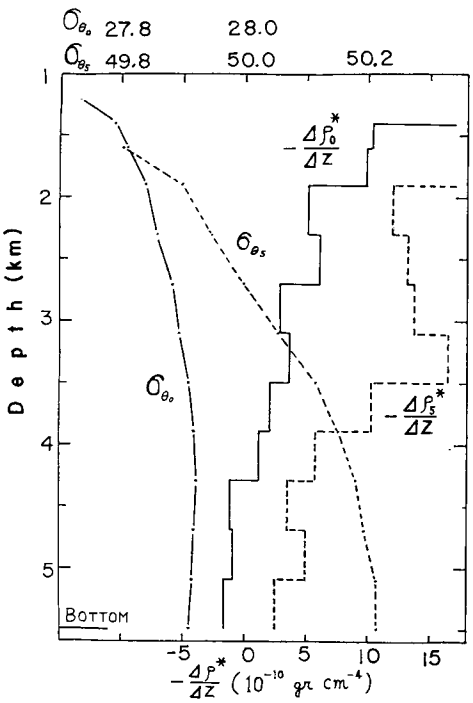

Fig. 1. Vertical distribution of potential density at St. Chain 159 (13 April 1960, 33°02'N, 56°30'W).
Subscripts $_0$ and $_5$ indicate respectively pressures of one atmosphere and 5000 decibars as the reference of potential temperature and density. $\sigma_{\theta_0}\equiv10^3(\rho_0^*-1)$ $\sigma_{\theta_5}\equiv10^3(\rho_5^*-1)$

5. Characteristics of a new potential vorticity

As shown in the previous Sections, the potential density ρ^* is a conservative quantity and its gradient is approximated well enough by a linear combination of two vectors $\nabla\rho$ and ∇p, so far as its vertical gradient is not minute. Since these

characteristics are nothing but two conditions (2) and (3), the potential density is what we are looking for to derive a conservative quantity concerning Ertel's vorticity theorem (1). Thus we obtain

$$\frac{DP}{Dt} \approx 0 \ , \tag{22}$$

where P is defined by

$$P \equiv -\frac{\boldsymbol{W} \cdot \nabla \rho^*}{\rho} \ . \tag{23}$$

The assumptions made in the course of deriving (22) are that sea water is not remarkably viscid and miscible and that the vertical gradient of potential density is not minute. These assumptions are not independent of each other. The portions of the sea, where the vertical gradient of potential density is minute and eddy viscosity and diffusion are large, are limited to the neighborhood of the sea surface and bottom. Accordingly, (22) holds for the sea except the surface and bottom frictional boundary layers and mixed layers.

Since the new quantity P has the same dimension of (vorticity/length) as the so-called potential vorticity Z/H and the latter is a kind of simplification of the former, we may conventionally call P a potential vorticity too.

Mathematically, Z/H is defined for a water column with some height, but P is for every point within the column. For a homogeneous layer of fluid, Z/H can be defined but P cannot be. This seems to be apparently a defect of P, but such a homogeneous layer is actually found only in the surface and bottom mixed layers, where any kind of potential vorticity cannot be conserved.

In deep water except in the vicinity of a strong current such as the Gulf Stream and Kuroshio and except in the vicinity of the equator, the scalar product of the horizontal components of $\nabla \rho^*$ and $\boldsymbol{W}$ is relatively small, and the vertical component of relative vorticity is much smaller than the Coriolis parameter. In this case, P is approximated by

$$P \approx -\frac{2\omega \sin\varphi}{\rho} \frac{\partial \rho^*}{\partial z} \ , \tag{24}$$

where ω is the angular speed of the earth's rotation, φ latitude positive in the Northern Hemisphere, and z a vertical coordinate positive upward.

We shall consider methods of numerical calculation of (23) in the following. To reduce errors coming from numerical differentiation, let us transfrom (23) into

$$P = -\frac{\nabla \cdot (\rho^* \boldsymbol{W})}{\rho} \ . \tag{25}$$

Based on the Gauss' theorem, its mean value for a bulk of water with a volume τ is given by

$$\bar{P}=\frac{\iiint P dxdydz}{\tau}=\frac{-1}{\bar{\rho}\tau}\iint(\rho^*Xdydz+\rho^*Ydzdx+\rho^*Zdxdy)\ , \tag{26}$$

where $\bar{\rho}$ is the mean density; and X, Y and Z are three components of absolute vorticity along coordinates axes. When x is at an angle ϵ to the east, y is directed to the left of x, and z is upward, they are given by

$$X=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}+2\omega\cos\varphi\sin\epsilon$$

$$Y=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}+2\omega\cos\varphi\cos\epsilon$$

$$Z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+2\omega\sin\varphi\ .$$

For shallow water above 1500 m, the error of approximation of

$$\rho_0^*\approx 1+10^{-3}\sigma_T$$

is less than an order of magnitude of 10^{-5} gr cm^{-3}. When currents are almost parallel and uniform along x axis, the downstream direction of the main current, and observation stations are occupied on a cross-current profile, eq. (26) is approximated by

$$\bar{P}\approx-\frac{10^{-3}}{\bar{\rho}\delta y}\left[\sigma_T\left(2\omega\cos\varphi\cos\epsilon+\frac{\delta u}{\delta z}\right)\right]_r^l$$

$$-\frac{10^{-3}}{\bar{\rho}\delta z}\left[\sigma_T\left(2\omega\sin\varphi-\frac{\delta u}{\delta y}\right)\right]_b^t\ , \tag{27}$$

where super- and subscripts beside the brackets have the same meaning as in the primitive of an integral, and l, r, t and b indicate respectively mean values on the left, right, top and bottom sides of a rectangle surrounded by two verticals at a couple of stations with a distance δy and two selected levels with a distance δz; and u indicates the current speed. The vertical component of current velocity and the latitudinal change in the Coriolis parameter are ignored in (27).

In deep water, except in the vicinity of a strong current and in the vicinity of the equator, one can easily calculate the potential vorticity based on (24).

Since the vertical shear of the geostrophic current is given by

$$\frac{\partial u}{\partial z}=\frac{g}{\rho 2\omega\sin\varphi}\frac{\partial\rho}{\partial y}\ ,$$

the ratio of the scalar product of the horizontal components of W and $\nabla\rho^*$ to the product of their vertical components becomes

$$R^*\equiv\left|\frac{W_h\cdot\nabla_h\rho^*}{Z\dfrac{\partial\rho^*}{\partial z}}\right|\equiv\left|\frac{X\dfrac{\partial\rho^*}{\partial x}+Y\dfrac{\partial\rho^*}{\partial y}}{Z\dfrac{\partial\rho^*}{\partial z}}\right|\approx\left|\frac{\rho\left(\dfrac{\partial u}{\partial z}\right)^2}{g\dfrac{\partial\rho}{\partial z}}\left(\frac{2\omega\sin\varphi}{Z}\right)\right| \tag{28}$$

for the following assumptions

$$X\approx 0,\ w\approx 0,\ \rho^*\approx\rho \text{ and } \left|\frac{\partial u}{\partial z}\right|\gg|2\omega\cos\varphi\cos\epsilon| .$$

If the vertical component of relative vorticity ζ is small, $2\omega\sin\varphi/Z$ becomes unity and we have

$$R^*\approx\frac{1}{R_i} , \tag{29}$$

where

$$R_i=\frac{-g\dfrac{\partial\rho}{\partial z}}{\rho\left(\dfrac{\partial u}{\partial z}\right)^2}$$

and is known as "Richardson's number" (Proudman [1953]).

Accordingly, Richardson's number might be a simplification of the ratio of a kind of the ordinary form of potential vorticity $-\frac{Z}{\rho}\frac{\partial\rho^*}{\partial z}$ to the additive terms in the new potential vorticity $-\frac{1}{\rho}\left(X\frac{\partial\rho^*}{\partial x}+Y\frac{\partial\rho^*}{\partial y}\right)$.

For shallow waters we have

$$Y\approx\frac{\partial u}{\partial z}=\frac{10^{-3}g}{\rho 2\omega\sin\varphi}\frac{\partial\sigma_T}{\partial y}$$

and the ratio becomes

$$R^*\approx\left|\frac{Y\dfrac{\partial\sigma_T}{\partial y}}{Z\dfrac{\partial\sigma_T}{\partial z}}\right|\approx\frac{10^{-3}g\left(\dfrac{\partial\sigma_T}{\partial y}\right)^2}{\left|\rho Z2\omega\sin\varphi\dfrac{\partial\sigma_T}{\partial z}\right|} .$$

A few examples are shown in Table 4, and one can see that the ordinary expression of potential vorticity would ignore terms of comparable magnitude to the ordinary potential vorticity itself in some limited regions with a relatively large value of anticyclonic horizontal shear. So far as we do not deal with detailed structure in the vicinity of the current axis, we may ignore the product of the horizontal components (Stommel [1955]).

Distributions of the potential vorticity in profiles across the Gulf Stream

Table 4. Ratios of $Y\partial\sigma_T/\partial y$ to $Z\partial\sigma_T/\partial z$

Region	Gulf Stream	Equatorial countercurrent	South equatorial current
Station	Atlantis I 4857–58	H.M. Smith 16 6—8	H.M. Smith 16 11—13
Latitude	38°N	6°N	3°N
Depth (m)	300	130	110
$\frac{\partial\sigma_T}{\partial y}\left(\frac{\text{sigma}}{\text{cm}}\right)$	0.4×10^{-6}	1.2×10^{-7}	-1.0×10^{-7}
$-\frac{\partial\sigma_T}{\partial z}\left(\frac{\text{sigma}}{\text{cm}}\right)$	0.2×10^{-4}	0.6×10^{-3}	0.5×10^{-3}
$\frac{\partial\sigma_T}{\partial y}\Big/\frac{\partial\sigma_T}{\partial z}$	2×10^{-2}	2×10^{-4}	2×10^{-4}
$2\omega\sin\varphi\ (\text{sec}^{-1})$	9.0×10^{-5}	1.5×10^{-5}	0.75×10^{-5}
$\zeta\ \ (\text{sec}^{-1})$	-3.0×10^{-5}	-0.5×10^{-5}	-0.5×10^{-5}
$Z=2\omega\sin\varphi+\zeta\ (\text{sec}^{-1})$	6.0×10^{-5}	1.0×10^{-5}	0.25×10^{-5}
$R^*=\left\|\frac{Y\partial\sigma_T/\partial y}{Z\partial\sigma_T/\partial z}\right\|$	1.5	0.2	1.1
References	Worthington [1954]	Austin [1954] Montgomery & Stroup [1962]	

and Kuroshio are not shown in the present paper. A conclusion to be reported here is that negative or slightly positive values of P are found in those same regions both in the Gulf Stream and in the Florida Current even though the vertical stability in those regions is large.

Since S and θ are conservative properties in the sea except in the mixed layers, so far they are used as two independent elements in water-mass analysis. Although $\rho^*\equiv\rho(S,\theta,p_r)$ is also a conservative property, it is completely determined by S and θ, and it will give the analysis no further information other than that its vertical gradient is used as a crude indicator of vertical stability by plotting $\sigma\theta$ curves on a θ-S plane. Now that we get another independent and conservative element P, it might become a useful tool for water-mass analysis, especially for that in the deep sea because of its simple expression as shown in (24).

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